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ON JAKOVLEV SPACES

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ABSTRACT

A topological space X is called weakly first countable, if for every point x there is a countable family $\{C_n^x \mid n \in \omega\}$ such that $x \in C_{n+1}^x \subseteq C_n^x$ and such that $U \subset X$ is open iff for each $x \in U$ some C_n^x is contained in U. This weakening of first countability is due to A. V. Arhangelskii from 1966, who asked whether compact weakly first countable spaces are first countable. In 1976, N. N. Jakovlev gave a negative answer under the assumption of continuum hypothesis. His result was strengthened by V. I. Malykhin in 1982, again under CH. In the present paper we construct various Jakovlev type spaces under the weaker assumption $\mathfrak{b} = \mathfrak{c}$, and also by forcing.

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0. Introduction

Motivated by a range of topological considerations and problems, Alexander Arhangelskii introduced the notion of weakly first-countable topological space in his 1966 paper "Mappings and spaces" ([2]), and posed there two fundamental questions:

Q1 Do the two notions, weakly first-countable and first-countable, coincide in the class of compact Hausdorff spaces?

Q2 Is there a Hausdorff, weakly first-countable compact space of cardinality $> \mathfrak{c}$?

Later, in 1969 [3], Arhangelskii proved his celebrated theorem that the size of compact Hausdorff first-countable space is at most the continuum, and it became evident that *yes* to the second question implies *no* to the first one.

Both of these questions are still open, although they have to be rephrased in light of newer consistency results.

Definition 0.1: A topological space $(X, \mathcal{T})$ is called **weakly first-countable**, if for each point $x \in X$ there is a countable family σ^x, $\sigma^x = \{C_n^x \mid n \in \omega\}$ with $x \in C_{n+1}^x \subseteq C_n^x \subseteq X$, such that

$$V \in \mathcal{T} \iff (\forall x \in V)\ (\exists n \in \omega)\ C_n^x \subseteq V.$$

Although, as it stands, the definition of *weakly first-countable* relies on the total *system* $\{\sigma^x \mid x \in X\}$, we will say that σ^x is a *local weak base* at x, and C_n^x are *weak neighborhoods* of x. In the following proposition we collect some basic facts about weakly first-countable spaces.

PROPOSITION 0.2:

(1) *Weakly first-countable spaces are sequential.*

(2) *Weakly first-countable Hausdorff spaces are first-countable iff Frechet-Urysohn.*

(3) *Compact Hausdorff sequential spaces of size $< \mathfrak{b}$ are Frechet-Urysohn. Hence compact Hausdorff weakly first-countable spaces of size $< \mathfrak{b}$ are first-countable.*

Proof: A space is sequential if any non-closed set A has a sequence whose convergence point is not in A. A space is Frechet-Urysohn if for every set A any point x in the closure of A has a sequence from A converging to x.

(1) If A is not closed, its complement contains a point x such that $A \cap C_n^x \neq \emptyset$ for all n. The key is to observe that any set meeting all members of a local weak base of a point in a weakly first-countable space has a countable subset

converging to this point. At this point we remark for the interested reader that weakly first-countable spaces are k-spaces, and that they are α_1 on the $\alpha_i (i = 1, 2, 3, 4)$ scale of A. Arhangelskii (defined in [4]).

(2) Suppose a Hausdorff space X is Frechet-Urysohn and weakly first-countable. We will show that $x \in \mathrm{Int}(C_n^x)$. If not, by Frechet-Urysohn, there is a sequence from $X \setminus C_n^x$ converging to x. Any point in C_n^x different from x has an open neighghborhood that is disjoint to the sequence. Together with C_n^x itself, we get an open neighborhood of x disjoint from the sequence.

(3) A proof is in [6]. (See also [7].) The main point is to prove that in a compact Hausdorff space of size $< \mathfrak{b}$ for every sequence of converging sequences with members in a set S and whose limits converge to some point p there is a sequence converging to p and consisting of elements taken from S. ∎

The first consistent example of a compact Hausdorff weakly first-countable not first-countable space — and so a consistent answer *no* to the first question of Arhangelskii — was given in 1976 by N. Jakovlev [9]. He constructed under CH such a space of cardinality $\aleph_1 = \mathfrak{c}$. (Non-compact examples are abundant, perhaps the simplest being Arens space of the "sequence of sequences" [1]).

After that, V. Malykhin ([11], Theorem 1) asserted that the CH construction of Jakovlev can be pushed up to any cardinal above the continuum $\mathfrak{c} = \aleph_1$ (we believe that the argument in [11] contains a gap, and we were able to reconstruct the proof only for $\aleph_2$). This gives a consistent *yes* to the second question of Arhangelskii.

The same paper [11] also contains a sketch (which we did not completely understand) of a forcing argument designed to add a Jakovlev type space of size $\aleph_1$, consistently with the negation of CH.

In the first section we define Jakovlev spaces (by Lemma 1.4, they are all counterexamples to Question Q1, and the example of Jakovlev [9] is a "Jakovlev space"), and in Theorem 1.5 we prove the existence of Jakovlev spaces of cardinality $\mathfrak{c}$ under the assumption $\mathfrak{b} = \mathfrak{c}$.

Under the same set-theoretic assumption, we construct in Theorem 1.6 a Jakovlev space of cardinality $\mathfrak{c}^+$.

By Proposition 0.2, under $\aleph_1 < \mathfrak{b}$ (just add $\aleph_2$ Hechler reals), there are no Jakovlev spaces of cardinality $\aleph_1$. However, we prove in Theorem 2.1 that Jakovlev spaces of size $\aleph_1$ exist in the extension of any universe by adding $\aleph_1$ Cohen reals (note that $\mathfrak{b} = \aleph_1$ there). This shows striking dissimilarity between weak first countability and first countability, because the size of a first countable compact space can be only countable or continuum.

Do Jakovlev spaces of size $\mathfrak{b}$ always exist? Not necessarily: Alan Dow has a model ([5], Theorem 4.6) in which $\aleph_1 = \mathfrak{b} < \aleph_2 = \mathfrak{s} = \mathfrak{a} = \mathfrak{c}$ and all compact sequential spaces of weight $\mathfrak{b}$ are Frechet-Urysohn, hence no Jakovlev spaces of size $\mathfrak{b}$. Together with our result, this gives consistency and independence of the existence of Jakovlev spaces of size $\mathfrak{b}$.

Is there a ZFC bound on the sizes of Jakovlev spaces? In section 3, using methods of [10], we construct a model in which there are Jakovlev spaces of any cardinality of the form $\kappa^{\aleph_0}$. This means that, consistently, there is no bound on the character of a point in a weakly first-countable space. The following two intriguing questions remain unanswered.

QUESTION 0.3: *Is yes to Question Q1 consistent? At least, is it consistent that there are no Jakovlev spaces?*

QUESTION 0.4: *Is it true in ZFC that there are arbitrarily large Jakovlev spaces?*

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1. Jakovlev spaces

We begin with definitions of pre-Jakovlev and Jakovlev spaces. As a preparation we define spaces of a specific form that we call J spaces.

Let κ be some ordinal, and L_n for $n \in \omega$ be disjoint copies of κ. We call L_n the n-th "line". For every $n \in \omega$, α_n: $L_n \to \kappa$ is a bijection which sets the correspondence between the n-th line and κ. We write $\alpha(p)$ rather than $\alpha_n(p)$ for $p \in L_n$ when the identity of n is obvious. So, for $p \in L_n$, $\alpha(p) \in \kappa$ is the "ordinal of p".

Definition 1.1: We say that a topological space with universe $L = \bigcup_{n\in\omega} L_n$ as above is a "J-space over L" if each point $p \in L$ has a countable neighborhood-base of compact, clopen, countable sets that "look down and to the left". By this we mean that there are clopen (closed and open) sets B^p_n for $n \in \omega$ such that

(1) $B_0^p \supseteq \cdots B_n^p \supseteq B_{n+1}^p \cdots$ form a neighborhood base of p, and
(2) if $p \in L_m$ and $x \in B_0^p$ then either $x = p$ or else $x \in L_k$ for some $k < m$ and $\alpha(x) < \alpha(p)$.
(3) Each B_n^p is compact and countable.

Thus a J-space is scattered, Hausdorff, first-countable, zero-dimensional, locally compact, and locally countable.

Now we define pre-Jakovlev spaces.

Definition 1.2: Let κ be some ordinal. We say that a topological space K is pre-Jakovlev built over κ iff the following hold: For some disjoint union $L = \bigcup_{n\in\omega} L_n$ of copies of κ, K is a J-space over L so that every infinite vertical set has an accumulation point. (A subset of L is said to be "vertical" iff it is a subset of some L_n.)

Definition 1.3: Let K be a pre-Jakovlev space built over some uncountable cardinal κ, and $L = \bigcup_{n\in\omega} L_n$ be its universe. The Jakovlev space $K^* = K \cup \{*\}$ is defined by the addition of a special point $*$ to L and the declaration that each open neighborhood of $*$ has the form $\{*\} \cup B$ where $B \subseteq L$ is open in K and includes for some $m \in \omega$ the set $\bigcup_{n\geq m} L_n$.

LEMMA 1.4: *If K is a pre-Jakovlev space built over an uncountable cardinal κ, and K^* is the Jakovlev space as defined above, then K^* is a compact, Hausdorff, zero-dimensional, weakly first-countable, not first-countable, scattered space of scattered height $\omega + 1$, and of sequential order 2.*

Proof: Since the pre-space K is Hausdorff, and as points in K have clopen neighborhoods, it is evident that K^* is Hausdorff. Indeed, the separation of any two points in K is assumed and the separation of $*$ from a point in K is a consequence of the assumption that points in K have clopen neighborhoods that lie entirely in the union of finitely many lines of K.

We next prove that every open neighborhood of the special point $*$ is co-countable. Let $G \subseteq K^*$ be an open neighborhood of $*$. Then by definition $G \cap K$ is open and includes almost all lines. We assume that the closed set $C = K \setminus G$ is uncountable and obtain a contradiction. Necessarily there is some $n \in \omega$ such that $C \cap L_n$ is uncountable, but there are only finitely such indices. Let n be the maximal one. Let A be the collection of accumulation points of $C \cap L_n$ in K. ($a \in A$ iff $a \in K$ and every neighborhood of a has an infinite intersection with $C \cap L_n$.) Since C is closed, A is a subset of C, and by property 2 every point of A lies in L_k for some $k > n$. It follows from the maximality of n

that A is countable. For every $a \in A$, B_0^a is an open and countable neighborhood of a. It follows that the uncountable set $(C \cap L_n) \setminus \bigcup_{a \in A} B_0^a$ has no accumulation point in K, which is a contradiction to our assumption.

We now argue for compactness of K^*. Since every open neighborhood of $*$ is co-countable, the space K^* is Lindelof. But every infinite set has an accumulation point, because if $X \subset K^*$ is infinite then either $X \cap L_m \neq \emptyset$ for infinitely many indices m (and then $*$ is an accumulation point of X), or else $X \cap L_n$ is infinite for some n (and then, by assumption, X has an accumulation point in K). A Lindelof space in which every infinite set has an accumulation point is compact. So K^* is compact.

An uncountable Jakovlev space is not first countable at the special point $*$. This follows directly from our observation that the complement of every open neighborhood of $*$ is countable, and so the intersection of countably many neighborhoods of $*$ is always infinite.

The space is weakly first-countable since the sets $\bigcup_{n \geq k} L_n$ form a local weak base for $*$. A moment's consideration convinces that the point $*$ has a local base of a particularly simple form

$$\mathcal{B} = \{K^* \setminus \bigcup_{p \in F} B_0^p \mid F \in [K]^{<\omega}\}.$$

Since $\mathcal{B}$ consists of clopen sets, K^* is zero-dimensional.

Observe that the first line L_1 consists of isolated points, but, for every $n > 1$, L_m contains unboundedly many non-isolated points for some $m \geq n$ (since every infinite vertical set has an accumulation point). It follows that K is scattered and of height $\omega + 1$. ∎

1.1. Jakovlev spaces of cardinality $\mathfrak{c}$, assuming $\mathfrak{b} = \mathfrak{c}$.

THEOREM 1.5: *Suppose that $\mathfrak{b} = \mathfrak{c}$. Then there exists a Jakovlev space $J_{\mathfrak{c}}^*$ of cardinality $\mathfrak{c}$.*

Proof: For functions $f, g \colon \omega \to \omega$ we denote by $f <^* g$ the eventual dominance relation: for some k for all $n \geq k$, $f(n) < g(n)$. Recall that $\mathfrak{b}$ is the least cardinality of a set of functions not dominated by any function. Assume $\mathfrak{b} = \mathfrak{c}$. By the previous lemma, it is enough to construct a pre-Jakovlev space J of cardinality $\mathfrak{c}$. Let L_n, for $n \in \omega$, be pairwise disjoint copies of $\mathfrak{c}$. Then $L = \bigcup_{n \in \omega} L_n$ is the universe of J. For $\xi < \mathfrak{c}$ let $L_n \restriction \xi$ be the set of first ξ members of L_n. That is, $L_n \restriction \xi = \{a \in L_n \mid \alpha(a) < \xi\}$. Define $L \restriction \xi = \bigcup_{n \in \omega} L_n \restriction \xi$. Fix an enumeration $\{F_\beta \mid \beta < \mathfrak{c}\}$ of all vertical sets of order type ω (in the ordering of each L_n induced by the correspondence α with $\mathfrak{c}$). We may assume that F_β is always a subset of $L \restriction \beta$. We shall define by induction on $\xi < \mathfrak{c}$ a topology on $L \restriction \xi$ by defining, for every point $p \in L \restriction \xi$, a neighborhood base $\langle B^p_i \mid i \in \omega\rangle$ consisting of compact, clopen, countable sets that form a J-space over $L \restriction \xi$. We let $J \restriction \xi$ be the topological space thus defined over $L \restriction \xi$. The final topology on L is determined by the local neighborhood bases thus defined. Observe that the neighborhood base of a point p in $J \restriction \xi_0$ remains the base of p in any of the spaces $J \restriction \xi$ for $\xi \geq \xi_0$. So for $\xi' < \xi$, $J \restriction \xi'$ is an open subspace of $J \restriction \xi$. Hence every compact subset of $J \restriction \xi'$ remains compact in $J \restriction \xi$, and in particular the neighborhood base of a point in $J \restriction \xi'$ consists of compact, clopen sets that remain compact, clopen in any later $J \restriction \xi$.

At limit stages ξ there is thus nothing to define, and the topological space $J \restriction \xi$ is the union of all spaces $J \restriction \xi'$ for $\xi' < \xi$.

Suppose that the bases for points in $J \restriction \xi$ are defined. We shall define the topological space $J \restriction \xi + 1$ by defining the neighborhoods of the points in $L \restriction \xi + 1 \setminus L \restriction \xi$. There are ω points in this level, formed by the ξ-th point taken from each L_n. Say $q_n \in L_n$ is the ξ-th point in L_n. With the possible exception of one point, all q_k shall be defined to be isolated (a point q is isolated when $\{q\}$ is open). The exception is determined by considering $F = F_\xi$. In case there is some n such that $F \subset L_n$ and F has no accumulation point in the space $J \restriction \xi$ so far defined, we define q_{n+1} as a limit of F in the following fashion.

Let $\langle f_i \mid i \in \omega\rangle$ be an increasing enumeration of F. Our intention is to find for each $i \in \omega$ a basic neighborhood $B^{f_i}_{k_i}$ of f_i such that the family $\{B^{f_i}_{k_i} \mid i \in \omega\}$ is discrete in $J \restriction \xi$. (That is, every $p \in J \restriction \xi$ has a neighborhood that intersects at most one $B^{f_i}_{k_i}$.) When we achieve this, we will make $q = q_{n+1}$ (a member of L_{n+1}) the one point compactification of this family by setting

$$B^q_m = \{q\} \cup \bigcup_{i \geq m} B^{f_i}_{k_i}.$$

The Hausdorff property of $J \restriction \xi + 1$ will follow from the discreteness of our family.

First we choose inductively m_i so that the basic neighborhoods $B^{f_i}_{m_i}$ are pairwise disjoint. (These can be easily obtained since each $B^{f_i}_{m_i}$ is clopen, and the f_i sequence is increasing.) Consider an arbitrary point $p \in (L \restriction \xi) \setminus F$. Since F has no accumulation points, we can fix some clopen neighborhood G_p of p such that $G_p \cap F = \emptyset$. We define a function $g_p \colon \omega \longrightarrow \omega$ by the requirement that for every $i \in \omega$,

$$G_p \cap B^{f_i}_{g_p(i)} = \emptyset.$$

The number $g_p(i)$ exists because f_i is not in G_p, and hence some set in the neighborhood basis of f_i is included in the complement of the closed set G_p. The collection $\{g_p \mid p \in (J \restriction \xi)\setminus F\}$ has cardinality $|\xi| < \mathfrak{c}$, and hence, by assumption $\mathfrak{b} = \mathfrak{c}$, there exists a function $g \in \omega^\omega$ such that $g_p <^* g$ for every $p \in (L \restriction \xi) \setminus F$. We may assume that $g(i) \geq m_i$ for every i, and hence the neighborhoods $\{B^{f_i}_{g(i)} \mid i \in \omega\}$ are pairwise disjoint. Now define the neighborhoods of $q = q_{n+1} \in L_{n+1}$ by the formula

$$B^q_m = \{q\} \cup \bigcup_{i \geq m} B^{f_i}_{g(i)}.$$

Each B^q_m is countable and open in $J \restriction \xi + 1$ (a union of open and countable sets), but it is also closed. Indeed, for every $p \in (L \restriction \xi) \setminus B^q_m$ there are two possibilities. If $p \in F$, then p is one of $f_0, \ldots, f_{m-1}$ (say $p = f_j$) and then $B^{f_j}_{g(j)}$ is disjoint from B^q_m (since the $B^{f_i}_{g(i)}$ are pairwise disjoint). If $p \notin F$, then G_p meets only finitely many $B^{f_i}_{g(i)}$ (since $g_p <^* g$) and hence $(G_p - B^q_m)$ is an open neighborhood of p disjoint from B^q_m.

This ends the definition of $J = \bigcup_{\xi < \mathfrak{c}} J \restriction \xi$. We argue that every infinite, vertical set has an accumulation point. It is enough to consider vertical sets of order-type ω. If $F \subset L_n$ is such a vertical set then $F = F_\xi$ for some ξ that is not smaller than the supremum of F. At stage ξ, F was considered. If it had an accumulation point at that stage, then that point remained an accumulation point. Otherwise, a limit point q was added in that stage.

Finally, the special point $*$ is added, completing the construction of the Jakovlev space $J^*_{\mathfrak{c}}$. ∎

1.2. Jakovlev spaces of cardinality $\mathfrak{c}^+$.

Theorem 1.6: *Assume $\mathfrak{b} = \mathfrak{c}$. Then there exists a compact, Hausdorff, weakly first-countable space of cardinality $\mathfrak{c}^+$. In fact there is a Jakovlev space of cardinality $\mathfrak{c}^+$.*

Proof: By Lemma 1.4, we only need to construct a pre-Jakovlev space of cardinality $\mathfrak{c}^+$. Let $\kappa = \mathfrak{c}^+$ be the successor of the continuum; the construction is carried out in κ stages, each stage consisting of $\mathfrak{c}$ steps. Let L_n be pairwise disjoint copies of κ for every $n \in \omega$. Fix a bijection α: $L_n \to \kappa$, and let q^ζ_n be the ζ-th member of L_n (that is $\alpha(q^\zeta_n) = \zeta$). As before, the space J is defined by assigning inductively for every q^ζ_n $(n \in \omega)$ clopen neighborhoods. The construction, however, is carried out somewhat differently.

(1) For every $\epsilon \in \kappa$ define $\gamma_\epsilon = \mathfrak{c} \cdot \epsilon$. Here $\mathfrak{c} \cdot \epsilon$ is the ordinal product, so that γ_ϵ is that ordinal consisting of ϵ copies of $\mathfrak{c}$.

(2) For every $\epsilon \in \kappa$ define the set of the ϵ stage:

$$K(\epsilon) = \{q_n^{\gamma_\epsilon+\xi} \mid \xi < \mathfrak{c} \text{ and } n \in \omega\}.$$

And, for $\zeta < \mathfrak{c}$, define $K(\epsilon) \restriction \zeta = \{q_n^{\gamma_\epsilon+\xi} \mid \xi < \zeta \text{ and } n \in \omega\}$. So $K(\epsilon)$ contains $\mathfrak{c}$ elements.

(3) For every $\xi < \kappa$ define $L_n \restriction \xi = \{a \in L_n \mid \alpha(a) < \xi\}$, and $J \restriction \xi = \bigcup_{n\in\omega} L_n \restriction \xi$. So $J \restriction \gamma_{\epsilon+1} = (J \restriction \gamma_\epsilon) \cup K(\epsilon)$ and, for limit ϵ,

$$J \restriction \gamma_\epsilon = \bigcup_{\epsilon'<\epsilon} J \restriction \gamma_{\epsilon'}.$$

The pre-Jakovlev space that we construct is $J = J \restriction \kappa = \bigcup_{\epsilon<\kappa} J \restriction \gamma_\epsilon$. The construction is by stages. In each stage $\epsilon < \kappa$ we define $K(\epsilon)$ in $\mathfrak{c}$ steps. At the ξ-th step, the neighborhoods of the points $q_n^{\gamma_\epsilon+\xi}$, $n \in \omega$ are defined. This construction is similar to the one in Theorem 1.5 (for cardinality $\mathfrak{c}$), and so we mainly describe the needed changes. We continue to denote by B_n^p the n-th neighborhood of p. It is, again, compact, clopen, and countable.

Since for limit ordinals ϵ, the topological space $J \restriction \gamma_\epsilon$ is defined as the union of the previous spaces $J \restriction \gamma_{\epsilon'}$ for $\epsilon' < \epsilon$, the inductive mission is to define the topology on $J \restriction \gamma_{\epsilon+1}$ from that on $J \restriction \gamma_\epsilon$.

We make two induction assumptions:

(1) The first inductive assumption is that for every *successor* ϵ the space $J \restriction \gamma_\epsilon$ is a pre-Jakovlev space.

It follows then that if ϵ is a limit ordinal in κ of uncountable cofinality, then $J \restriction \gamma_\epsilon$ is also a pre-Jakovlev space. The main point is that $J \restriction \gamma_\epsilon$ is the union of previously constructed $J \restriction \gamma_{\epsilon'}$ for $\epsilon' < \epsilon$, and so every countably infinite vertical set is included in some $J \restriction \gamma_{\epsilon'}$ for ϵ' a successor ordinal.

In case ϵ is a limit ordinal of countable cofinality, $J \restriction \gamma_\epsilon$ is a J-space, but obviously not a pre-Jakovlev space, since cofinal sequences have no accumulation points. This will be taken care of at the next stage of the construction (which endures $\mathfrak{c}$ steps, and is of the same character as the construction for Theorem 1.5).

(2) The second inductive assumption is that if $p \in K(\epsilon)$, then for every $\epsilon' < \epsilon$ there is $n \in \omega$ such that $B_n^p \cap J \restriction \gamma_{\epsilon'+1} = \emptyset$

If $\epsilon < \kappa$ is a successor ordinal or a limit ordinal of uncountable cofinality (or zero), then the topology of $K(\epsilon)$ is defined as in the previous section as a Jakovlev space of cardinality $\mathfrak{c}$, and then $J \restriction \gamma_{\epsilon+1} = J \restriction \gamma_\epsilon \cup K(\epsilon)$.

So suppose that $\epsilon < \kappa$ is a limit ordinal of countable cofinality and $J \restriction \gamma_\epsilon = \bigcup_{\epsilon'<\epsilon} J \restriction \gamma_{\epsilon'}$ has been defined. We must define the topology on $J \restriction \gamma_{\epsilon+1}$ by defining the clopen neighborhoods B^p_n for points $p \in K(\epsilon) = J \restriction \gamma_{\epsilon+1} \setminus J \restriction \gamma_\epsilon$. We fix an enumeration $\{F_\beta \mid \beta < \mathfrak{c}\}$ of all countable, vertical subsets of $J \restriction \gamma_{\epsilon+1}$. But actually we are only interested in two types of such sets.

(1) A countable, infinite vertical set $F \subset L_n$ of order-type ω is "of the first type" iff F is an unbounded subset of $L_n \restriction \gamma_\epsilon$ of order-type ω in the well-ordering of L_n.

(2) F is "of the second type" iff it is included in $K(\epsilon)$.

We are not interested in infinite vertical sets that are bounded in γ_ϵ because they are already in some $J \restriction \epsilon'$ for a successor $\epsilon' < \epsilon$ and thus have an accumulation point by the inductive assumption.

Every $q \in K(\epsilon)$ has the form $q_n^{\gamma_\epsilon+\xi}$, for some $\xi < \mathfrak{c}$, and the definition of the neighborhoods of q is by induction on ξ. Suppose that $J \restriction \gamma_\epsilon \cup K(\epsilon) \restriction \xi$ has been defined. At step ξ we define the neighborhoods of the ω points at level ξ, namely $q_n^{\gamma_\epsilon+\xi}$ for $n \in \omega$. For this we consider the set F_ξ enumerated at this step. In case F_ξ is of the second type, the construction proceeds just as in the Jakovlev construction of the previous section.

A slightly more complex case is when F_ξ is of the first type. The problem is that $J \restriction \gamma_\epsilon$ may already have size $\mathfrak{c}$, so that it is not possible to dominate all the functions that its points define. Fortunately there is no need for this, by the second inductive property.

How does this help? Suppose that $F = F_\xi$ is of the first type, and $F \subset L_n$ has no accumulation point in the space $J \restriction \gamma_\epsilon + \xi$ so far constructed. Enumerate $F = \{f_i \mid i \in \omega\}$ in increasing order (i.e., $\alpha(f_i) < \alpha(f_{i+1})$). By diluting F we may assume (for simplicity of expression) that each f_i lies in a separate $K(\epsilon')$. That is, $f_i \in K(\epsilon_i)$ where $\epsilon_i < \epsilon_{i+1}$. Choose $j(i) \in \omega$ such that the neighborhoods $B^{f_{i+1}}_{j(i+1)} \cap J \restriction \gamma_{\epsilon_i} = \emptyset$. It is here that the second inductive hypothesis is used. Since the sequence F is unbounded in γ_ϵ, we are sure that the family $\{B^{f_i}_{j(i)} \mid i \in \omega\}$ is discrete with respect to the points in $J \restriction \gamma_\epsilon$. But the collection of remaining points in $K(\epsilon) \restriction \xi$ is a set of cardinality $< \mathfrak{c}$, and hence the assumption that $\mathfrak{b} = \mathfrak{c}$ can be used as in the proof of Theorem 1.5 to refine the neighborhoods $\{B^{f_i}_{j(i)} \mid i \in \omega\}$ and to obtain a family discrete in $J \restriction \gamma_\epsilon \cup K(\epsilon) \restriction \xi$, as required. ∎

2. A Jakovlev space of cardinality $\aleph_1 < 2^{\aleph_0}$

In this section we show the consistency of $2^{\aleph_0} > \aleph_1$ with the existence of a Jakovlev space of cardinality $\aleph_1$. We prove that such a space exists in any extension of the universe obtained by adding (at least) ω_1 Cohen reals.

THEOREM 2.1: *If $\overline{r} = \langle r_\alpha \mid \alpha < \omega_1 \rangle$ is a generic sequence of ω_1 Cohen reals over the ground model V, then in $V[\overline{r}]$ there exists a Jakovlev space of cardinality $\aleph_1$.*

Proof: We work in $V[\overline{r}]$. The idea is this. The space J is constructed as in section 1.1, and we use the same terminology used there with ω_1 replacing $\mathfrak{c}$. So the universe of J is $L = \bigcup_{n\in\omega} L_n$ where L_n is a copy of ω_1. We intend to construct a pre-Jakovlev space by defining for every point $p \in L$ a countable base B^p_n, $n \in \omega$, consisting of compact, clopen, countable sets that satisfy the properties of a pre-Jakovlev space. But now there is no enumeration of all possible countable vertical sets, since the continuum may be larger than ω_1. Instead, we let the generic real r_α choose the omega sequence at the α-th step of the construction. Since r_α is generic, we will be able to prove that every countable vertical set has an accumulation point.

Fix two functions, $\beta\colon \omega_1 \to \omega_1$ (with $\beta(\xi) \leq \xi$) and $n\colon \omega_1 \to \omega$, such that for every $\beta_0 \in \omega_1$ and $n_0 \in \omega$ there are uncountably many ordinals $\xi \in \omega_1$ with $\beta(\xi) = \beta_0$ and $n(\xi) = n_0$. At step $\xi < \omega_1$ of the construction, we assume that $J \restriction \xi = \bigcup_{n\in\omega} L_n \restriction \xi$ is defined. That is, we assume that for every p with $\alpha(p) < \xi$ neighborhoods B^p_n that satisfy properties (1)–(3) of J-spaces have been defined. In particular, B^p_n is a clopen, compact, and countable neighborhood of p. Consider $\beta_0 = \beta(\xi)$ and $n_0 = n(\xi)$. We shall try to define an omega sequence $F \subset L_{n_0} \restriction \beta_0$ that has no accumulation points in J_ξ.

For this we define a forcing poset P as follows. A condition $p \in P$ has the form $p = (\langle b_0, \ldots, b_{k-1}\rangle, X)$ where $b^p = \langle b_0, \ldots, b_{k-1}\rangle$ is a finite sequence such that its members b_i all lie in $L_{n_0} \restriction \beta_0$ and are enumerated in the increasing order (i.e., $\alpha(b_0) < \alpha(b_1) < \cdots < \alpha(b_{k-1})$), and $X^p = X$ is a finite subset of $J \restriction \xi$. As for the ordering, we define $p_1 \leq p_2$ (read: p_2 extends p_1) iff b^{p_2} is an end-extension of b^{p_1} (say $b^{p_2} = b^{p_1} {}^\frown \langle b_k, \ldots, b_{m-1}\rangle$), $X^{p_1} \subseteq X^{p_2}$, and for every $x \in X^{p_1}$, $\{b_k, \ldots, b_{m-1}\} \cap B^x_0 = \emptyset$. That is, a condition p_1 ensures that the continuation of its ω sequence is disjoint from the first basic neighborhood of each point in X^{p_1}.

It is easily checked that $p_1 \leq p_2$ is a partial ordering on P, and that any condition can be extended to include in X arbitrary $x \in J \restriction \xi$.

It may well be the case that some condition p has no extensions that continue the b^p sequence. If this happens then $X = X^p$ has the property that $\bigcup\{B_0^x \mid x \in X\}$ covers a final segment of $L_{n_0} \restriction \beta_0$. In this case we should not worry. Since each B_0^x is compact, every infinite sequence from B_0^x has an accumulation point and hence every sequence unbounded in $L_{n_0} \restriction \beta_0$ has an accumulation point.

So assume that for every finite set $X \subset J \restriction \xi$, $L_{n_0} \restriction \beta_0 \setminus \bigcup\{B_0^x \mid x \in X\}$ is unbounded in $L_{n_0} \restriction \beta_0$. In this case any generic filter G produces an increasing ω sequence, $F_G = \langle b_0, b_1, \ldots \rangle$, of members of $L_{n_0} \restriction \beta_0$ that has finite intersection with any B_0^p for $p \in J \restriction \xi$. So F_G has no accumulation points in $J \restriction \xi$. Obviously, the poset P is countable, and is thence isomorphic to the Cohen forcing for adding a real. Thus r_ξ can be viewed as $V[\langle r_\alpha \mid \alpha < \xi \rangle]$ generic, yielding a generic sequence R_ξ ($= F_G$ above). The construction continues as in Section 1 by defining a discrete family of compact neighborhoods of the points of R_ξ, and using them to define a neighborhood base for $q^\xi_{n_0+1}$. (The fact that $J \restriction \xi$ is countable is used here to find a function that dominates countably many functions.)

Now we must prove that every infinite vertical subset of J has an accumulation point. Suppose that $A \subset L_{n_0}$ is such a set (of order-type ω) and let $\beta_0 = \sup A$. There is some ordinal $\xi < \omega_1$ such that $A \in V[\langle r_{\xi'} \mid \xi' < \xi \rangle]$, and such that $\beta(\xi) = \beta_0$ and $n(\xi) = n_0$. If a final segment of $L_{n_0} \restriction \beta_0$ is covered by some finite union of basic neighborhoods B_0^p ($p \in J \restriction \xi$), then A has an accumulation point in $J \restriction \xi$, being almost included in a finite union of compact sets. Otherwise, a density argument shows that the generic ω sequence constructed at the ξ-th step intersects A infinitely often. Thus $q^\xi_{n_0+1}$ is an accumulation point of A. ∎

3. A model with arbitrarily large Jakovlev spaces

We prove in this section that if $V[G]$ is any generic extension of V obtained by iterating an increasing sequence of dominating reals, then there are in $V[G]$ Jakovlev spaces with arbitrary large cardinalities. In fact, if κ is any cardinal such that $\kappa^{\aleph_0} = \kappa$ holds in $V[G]$, then there is in $V[G]$ a Jakovlev space of cardinality κ.

The poset P for adding a dominating real is the well-known poset devised by Hechler ([8]). A condition (f, H) in P consists of a finite function f from ω to ω and a finite collection $H \subset \omega^\omega$ of total functions from ω to ω. The extension relation is defined naturally by (f_1, H_1) extending (f_0, H_0) iff $f_0 \subseteq f_1$, $H_0 \subseteq H_1$, and for every $g \in H_0$, $g(k) \leq f_1(k)$ for every $k \in \mathrm{dom}(f_1) \setminus \mathrm{dom}(f_0)$. Standard properties of P are that it satisfies the c.c.c. and that a generic filter G produces a new real $g = \bigcup\{g \mid \exists H(g, H) \in G\}$, which dominates every ground model real in the eventual dominance relation $\leq^*$.

Define a finite support iteration P_α, $\alpha < \omega_1$, of such Hechler forcing posets, and let G be a V generic filter over P_{ω_1}. Let $\langle r_\alpha \mid \alpha \in \omega_1\rangle$ be the sequence of generic reals produced by G. P_{ω_1} satisfies the c.c.c. and its cardinality is $\mathfrak{c}$.

THEOREM 3.1: *The following holds in $V[G]$ (a generic extension obtained by iteratively adding ω_1 Hechler reals). If κ is any cardinal such that $\kappa^{\aleph_0} = \kappa$ then there is a Jakovlev space of cardinality κ.*

Proof: For $n \in \omega$ and $\alpha \leq \omega_1$, let $L_n(\alpha) = \kappa \cdot \alpha \times \{n\}$ and $J(\alpha) = \bigcup_{n\in\omega} L_n(\alpha)$. τ_α will denote some basis for a zero-dimensional topology on $J(\alpha)$. We will construct a Jakovlev space $J = (J(\omega_1), \tau_{\omega_1})$ by induction in ω_1 stages. For α limit we define $\tau_\alpha = \bigcup_{\beta<\alpha} \tau_\beta$.

Assume that at the stage $\alpha < \omega_1$ we have τ_α on $J(\alpha)$ with $\tau_\alpha \subset V^{P_\alpha} = V[\langle r_\beta \mid \beta < \alpha\rangle]$. We also assume (identifying topology with its basis) that $(\forall\gamma < \alpha)$ $\tau_\alpha \upharpoonright J(\gamma) = \tau_\gamma$, i.e., that the subspaces $J(\beta)$ of J are constructed open, and remain open and unchanged thereafter.

Write $\alpha = \beta + n$, with β a limit ordinal and $n \in \omega$.

Working in V^{P_α}, consider a maximal almost disjoint family, say $\mathcal{A}$, of countable discrete in $(J(\alpha), \tau_\alpha)$ subsets of $L_n(\alpha)$. We can write $\mathcal{A} = \{A_\xi : \xi \in \kappa\}$ because $\kappa^{\aleph_0} = \kappa$ holds even in V. With the help of the V^{P_α}-generic dominating real r_α we will make each point $p_\xi =< \kappa \cdot \alpha + \xi, n+1 >$ the sequential limit of the index-corresponding set A_ξ, the rest of the points of $J(\alpha+1) \setminus J(\alpha)$ declared isolated.

Fix $\xi < \kappa$. Write $A = A_\xi = \{a_i : i \in \omega\}$.

As before we denote by B^x_n the n-th neighborhood of x, which is countable, compact, and looking downward and to the left, as in the definition of J-spaces.

Pass to a larger model $V^{P_{\alpha+1}}$ and consider $\mathcal{B} = \{B^{a_i}_{r_\alpha(i)} : i \in \omega\}$.

Define a local base of p_ξ by setting

$$B^{p_\xi}_n = \{p_\xi\} \cup \bigcup\{B^{a_i}_{r_\alpha(i)} : i \geq n\}.$$

Now unfix ξ and, using the same dominating real r_α, do exactly the same for every $\xi \in \kappa$. This ends the description of the induction stage α.

Next, we show, now by induction on $\xi < \kappa$, that this short construction works.

For clarity, we single out the following simple topological fact:

if X is a regular space and A is a countable subspace, then a point p can be adjoined to X with A converging to p in the regular space $X \cup \{p\}$ if and only if every point a_i of A has an open neighbourhood N_i such that $\mathcal{N} = \{N_i : i \in \omega\}$ is a locally finite family in the space X. (And then, in the direction of interest to us, a countable base for p can be defined as $\{p \cup \bigcup_{i \geq n} N_i : n \in \omega\}$.)

By this remark with $X = J(\alpha) \cup \{p_\eta : \eta < \xi\}$, it is sufficient to show that $\mathcal{B}$ is a locally finite family in the space $J(\alpha) \cup \{p_\eta : \eta < \xi\}$.

First, consider a point $x \in J(\alpha) \setminus A$. Since no $x \in J(\alpha)$ is a limit point of A in τ_α, for every x not in A there is a function $f_x \in {}^\omega\omega$ determining an open neighbourhood of A whose closure does not contain x, namely:

$$x \notin \left(\overline{\bigcup_{i \in \omega} B^{a_i}_{f_x(i)}}\right).$$

Let C be a compact open neighbourhood of x disjoint from $\bigcup_{i \in \omega} B^{a(i)}_{f_x(i)}$. Since r_α dominates ${}^\omega\omega \cap V^{P_\alpha}$, there is a number $k \in \omega$ such that

$$C \cap \overline{\bigcup \mathcal{B}} \subseteq B^{a_0}_{r_\alpha(0)} \cup \cdots \cup B^{a_k}_{r_\alpha(k)},$$

and hence this C witnesses the local finiteness of $\mathcal{B}$ at x.

Next, since A is closed discrete in $J(\alpha)$, $\exists f \in {}^\omega\omega \cap V^{P_\alpha}$ such that the family $\{B^{a_j}_{f(j)} : j \in \omega\}$ is pairwise disjoint, and hence discrete at all points $x \in A$. Since ${r_\alpha}^* \geq f$, our $\mathcal{B}$ is locally finite at the points $x \in A$.

Lastly, consider p_η with $\eta < \xi$. Write $A_\eta = \{a^\eta_i : i < \omega\}$. Since $| A_\eta \cap A | < \omega$, $\exists n_0 \in \omega$ such that $\{a^\eta_i : i \geq n_0\}$ and A are disjoint closed discrete sets in $J(\alpha)$. Therefore, $\exists f \in {}^\omega\omega$ such that $\bigcup\{B^{a^\eta_i}_{f(i)} : i \geq n_0\} \bigcap \bigcup\{B^{a_i}_{f(i)} : i \in \omega\} = \emptyset$. Again by ${r_\alpha}^* \geq f$, $\exists n_1 \geq n_0$ such that $\bigcup\{B^{a^\eta_i}_{r_\alpha(i)} : i \geq n_1\} \bigcap \bigcup\{\{B^{a_i}_{r_\alpha(i)} : i \geq n_1\} = \emptyset$, and so $\{p_\eta\} \cup \bigcup\{B^{a^\eta_i}_{r_\alpha(i)} : i \geq n_1\}$ is a neighbourhood of p_η witnessing local finiteness of $\mathcal{B}$ at p_η. This ends the proof of correctness of the inductive stage α.

Finally, as does any countable set of ordinals, every countable vertical set in L_n in the final model $V[\langle r_\zeta \mid \zeta < \omega_1 \rangle]$ is already in some $V[\langle r_\zeta \mid \zeta < \alpha \rangle]$, $\alpha < \omega_1$, and hence acquired a limit point during induction. Therefore $J(\omega_1)$ is a pre-Jakovlev space in $V[G]$ of cardinality κ as required. ∎

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